Strongly damped Stiff Oscillator under External Force: A Case Study

Winfried Auzinger[1], Roksolyana Stolyarchuk[2], Roman Stoliarchuk[3]

1. Institute of Analysis and Scientific Computing, TU Wien, AUSTRIA,Vienna, Wiedner Hauptstrasse 8-10. Email: w .auzinger@tuwien.ac.at
2. IInstitute of Applied Mathematics and Fundamental Sciences, Lviv Polytechnic National University, UKRAINE, Lviv, 12 Bandery str. Email: sroksolyana@yahoo.com
3. Institute of Telecomunications, Radioelectronics and Electronic Engineering, Lviv Polytechnic National University, UKRAINE, Lviv, 2 Profesorska str. Email: romanstl1208@gmail.com

Abstract – This is a numerical study of ODE systems modelling the dynamics of a strongly damped stiff oscillator under periodic force. After reviewing the notion of stiffness, we demonstrate and discuss convergence order reduction for some well-established numerical integrators. We also include some comments on stiffness in nonlinear systems.

Keywords – harmonic oscillator, stiffness, strong damping, numerical integration, order reduction, stiffness and nonlinearity

Introduction

Typical phenomena which are well known in the context of the numerical integration of initial values problems for ODEs are instability effects and reduced convergence orders when standard integration methods are applied to stiff problems. Originally, the term *stiffness* was introduced by G.Dahlquist (see for instance [4]); it means that general, transient solutions rapidly converge towards a quasi-stationary, smooth solution; this is asymptotically a very stable situation, but it poses a significant challenge for numerical integration. One of the earliest works on the effect of stiffness in numerical integration is [7], where scalar linear ODEs of first order were considered, and order reduction effects occurring in numerical methods were demonstrated. Apart from this effect, the scalar case is special in the sense that the difference between a transient and a smooth solution decays monotonously. Within the past decades, the theory of numerical integration of stiff problems has made significant progress, see for instance [5] and references therein.

A general stability and convergence analysis of numerical methods relies on particular a priori assumptions about the given problem. Consider for instance a model problem in form of a 2 x 2 linear ODE system

$$y'(t) = A\,.\,y(t),$$

where the real-valued matrix $A = A(\varepsilon)$ is diagonalizable, with eigenvalues $\xi = O(1) =$ const. and $\eta = \eta(\varepsilon) = -O(1/\varepsilon)$, with $0 < \varepsilon << 1$.

A rigorous and sharp estimate for the (worst-case) local growth of transient solutions $y(t)$, measured in the Euclidean norm $\|\cdot\|$, is given by ([5])

$$\| y(t) \| \;\leq\; \exp(\mu(A)\, t) \cdot \| y(0) \|,$$

where $\mu(A)$ denotes the so-called logarithmic norm of A, i.e., the right-most eigenvalue of the symmetric matrix $(A+A^*)/2$. (Here, A^* denotes the transpose of A.)

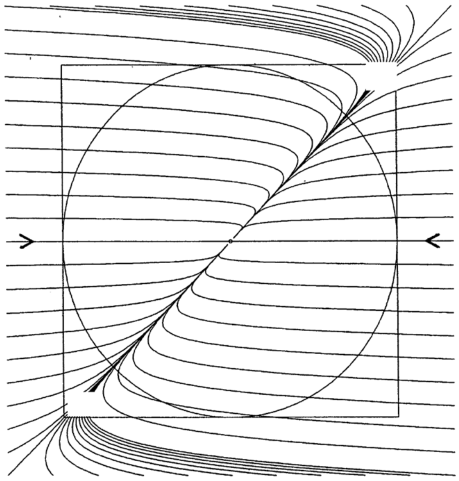

Fig. 1. Phase portrait of a stiff 2x2 system (cited from [2])

In [2] it was shown that, in general. (note the + sign!)

$$\mu(A) = +O(1/\varepsilon) \quad \text{for} \quad \varepsilon \to 0,$$

if the matrix A is not normal (i.e., not symmetric,). Only for the symmetric case $A = A^*$ we have $\mu(A) = ((A+A^*)/2) = \xi = O(1)$ independent of ε. For a visualization of this effect see Fig. 1: Here, despite $\xi = 0$ we have $\mu(A) \gg 0$, and the norm $\| y(t) \|$ may grow locally very fast in the horizontal direction prior to damping, unless the eigendirections are orthogonal (which is equivalent to symmetry, $A = A^*$).

In the present paper we study a stiff linear model ODE system of a special type, involving what may be called an even more severe non-normality effect in the sense of [8] (see also [2]). In particular, the problem Eq. (1) resp. Eq. (4) considered below also has asymptotically stable solutions which also may locally grow very fast over small transient time intervals. For such a stiff model problem we study and compare the convergence behavior of two different standard second order integration methods.

Strongly Damped Stiff Oscillator

The dynamics of the time-dependent deflection $u(t)$ of a (free) damped harmonic oscillator with eigenfrequency $\omega > 0$ and damping parameter $\rho > 0$ is described by the second-order ordinary differential equation

$$u''(t) + 2\,\rho\, u'(t) + \omega^2\, u(t) = 0 \quad (t > 0)\,, \qquad (1)$$

subject to initial conditions

$$u(0) = a, \quad u'(0) = b. \qquad (2)$$

This is a simple model problem; however here we consider the case of a stiff oscillator subject to strong damping, i.e.,

$$\omega \gg 0, \quad \rho \gg 0, \qquad (3)$$

and we are interested in the performance of numerical integration methods applied to a system of such a type. We further assume that it is subject to an additional strong external periodic force. In particular, we consider the 'most critical' case $\rho = \omega$, and consider as a test model the inhomogeneous linear second-order differential equation

$$u''(t) + 2\,\omega\, u'(t) + \omega^2\,(u(t) - \cos(t)) = 0. \qquad (4)$$

Remark: The choice $\rho = \omega$ simplifies some technicalities to follow; however this is not essential, and more generally, similar numerical effects as reported in the following are observed.

From an analytical point of view, the problem Eq. (4) is of course simple to understand. Asymptotically for $t \to \infty$, the solutions to Eq. (4) behave in a very stable way: All solutions $u(t)$ for arbitrary initial values given by Eq. (2) rapidly converge to a particular, 'damped-out', smooth solution

$$U(t) \;=\; (\omega/(\omega^2 + 1))^2 \;.\; ((\omega^2 - 1)\cos(t) + 2\,\omega\,\sin(t)), \qquad (5)$$

where $U(0) \in [0,1)$, and the limit of $U(t)$ for $t \to \infty$ is contained in the interval $[-(5+3\sqrt{3})/8,+(5+3\sqrt{3})/8]$. More precisely: Since $-\omega$ is a double root of the characteristic polynomial

$$\chi(\lambda) \;=\lambda^2 + 2\,\omega\,\lambda + \omega^2$$

of the homogeneous problem, the general solution to Eq. (4) is given by

$$u(t) = U(t) + c\,\exp(-\omega\, t) + d\, t\, \exp(-\omega\, t), \qquad (6)$$

where the constants c and d are uniquely determined in terms of the given initial values a and b from Eq. (2). The derivative of $u(t)$ is

$$u'(t) = U'(t) - c\,\omega\,\exp(-\omega\, t) + d\,(1 - \omega\, t)\,\exp(-\omega\, t). \qquad (7)$$

Note that for $d \neq 0$ the convergence of $u(t)$ towards $U(t)$ is not monotonic in general. In fact, what occurs is an effect called ‚hump', namely a transient growth locally near $t = 0$ prior to asymptotic damping. This means that, while the given system behaves asymptotically stable for $t \to \infty$, it locally behaves unstable, with a transient blow-up over small times. Fig. 2 shows a purely qualitative visualization of this effect.

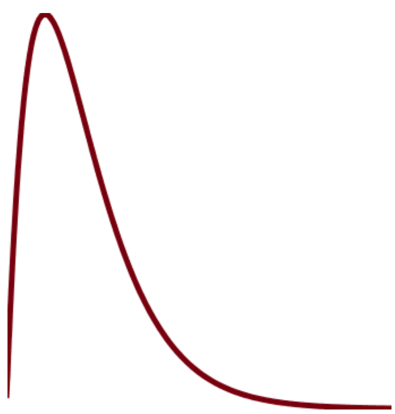

Fig. 2. The hump (cf. Fig. 1)

However we may ask what, in fact, is a natural measure for the growth or decay of solutions. In physical terms one may argue that for the homogeneous problem Eq. (1) a natural measure is given by the total energy

$$E(u,u') = \sqrt{(\omega^2 u^2 + (u')^2)}, \qquad (8)$$

which is decreasing for increasing t, i.e.,

$$\mathrm{d}/\mathrm{d}t \; E(u(t),u'(t)) < 0,$$

see [1],[2].

Standard Numerical Integrators

(In the following we do not specify all details concerning standard numerical integration methods in detail, as this can be found in the standard literature as for instance [5].)

A problem of the type considered here shows a stiff behavior. As mentioned before, the existence of strongly transient and asymptotically damped solutions $u(t)$ poses a challenge for conventional methods like for instance explicit Runge-Kutta, because these methods behave in an unstable way and may not properly converge, even if the integration simply tries to follows a smooth, non-transient solution $U(t)$.

In principle, the remedy to this problem is well-known: For a stable numerical integration, appropriate *implicit* methods are necessarily required. 'Implicit' means that in each discrete integration step $t \to t + \Delta t$ with a given stepsize Δt, the new approximation for $u(t + \Delta t)$ is to be obtained by solving a system of implicit equations in terms of the previous approximation to $u(t)$, which requires some additional computational effort compared to straightforward explicit integration methods.

The simplest stable implicit integration schemes for stiff ODE systems of first order are the so-called Implicit Euler Scheme (IES) and the Implicit Midpoint Scheme (IMS), see e.g. [5]. We now are apply these schemes to problem Eq. (4). To this end we use a common transformation, namely defining

$$v = u' \qquad (9)$$

as a dependent variable, and we consider the equivalent two-dimensional first-order ODE system in the variables u and v,

$$\begin{aligned} u'(t) &= v(t), \\ v'(t) &= -\omega^2 u(t) - 2\,\omega\, v(t) + \omega^2 \cos(t). \end{aligned} \qquad (10)$$

The coefficient matrix A of this system is not diagonalizable, with a double negative eigenvalue $-\omega$. Nevertheless, its logarithmic norm is very large, namely

$$\mu(A) = +O(\omega^2)\ (!) \quad \text{for } \omega \to \infty, \quad \text{see [1]}.$$

In the classical, conventional sense, IES has an asymptotical convergence order $p = 1$, i.e., its global error is proportional to Δt for $\Delta t \to 0$. IMS has asymptotical convergence order $p = 2$, i.e., its error is proportional to $(\Delta t)^2$.

Another popular numerical method of convergence order $p = 2$ is the so-called 2-step Backward Differentiation Scheme (BDF2), a two-step generalization of IES, see [5].

All these methods are A-stable, i.e., they behave in a stable way for $t \to \infty$ when applied to an arbitrarily stiff scalar problem. In contrast to IMS, the BDF2 method is even strongly A-stable. (See [5] for a precise description of these notions of stability.)

Numerical Investigation of Problem from Eqs. (4), (10)

We now perform a numerical study of the IMS and BDF2 schemes applied to problem (10), where the parameter ω is chosen as ω = 1E+5. In particular, we study the actual behavior of the approximation error in dependence of the stepsize Δt, starting at $U(0)$ from Eq. (5).

We stress that the fact that solutions of Eq. (4) or Eq. (10), respectively, show a transient behavior as visualized in Fig. 2 is the essential difficulty here.

Tables 1 and 2 show the results for both numerical methods considered. Problem Eq. (10) was integrated from $t = 0$ up to $t = 10$, first using 100 steps with $\Delta t = 0.1$, then using 200 steps with $\Delta t = 0.05$. The approximation errors at $t = 10$ in u, u', as well as in the energy $E(u,u')$ (see Eq. (8)) are specified together with the actual order p of the error observed on step halving from $\Delta t = 0.10$ to $\Delta t = 0.05$.

Table 1: Numerical results for IMS

Δt	error in $\lvert u\rvert$	$\lvert$error in $u'\rvert$	error in energy
0.10	2.2E-3	4.8E+0	2.2E+2
0.05	4.9E-4 (p=2.2)	4.3E+0 (p=0.2)	4.9E+1 (p=2.2)

Table 2: Numerical results for BDF2

Δt	error in $\lvert u\rvert$	$\lvert$error in $u'\rvert$	error in energy
0.10	3.2E-8	1.6E-3	3.6E-3
0.05	8.5E-9 (p=1.9)	4.3E-4 (p=1.9)	9.4E-4 (p=1.9)

From Tables 1 and 2 one can see that the convergence order in u and with respect to the energy norm $E(u,u')$ (see Eq. (8)) is $p \approx 2$. The strongly stable BDF2 method is much more accurate than IMS, by a factor approximately ω = 1E+5. Concerning u', the approximation quality provided by IMS is *stagnant:* The observed order p is close to 0, which is an extreme example of order reduction. Similar order reduction effects are also observed for higher order methods and other types of methods like for instance exponential integrators (see [6]), and also if these are applied to certain types of linear or nonlinear stiff ODE systems, as for instance Example from Eq. (11) below which we briefly discuss in the Appendix.

Appendix: A Nonlinear Stiff System

We conclude our discussion of stiffness effects by considering an example of a nonlinear system (sometimes called the '*Vienna problem*') which was considered in more detail in [9]:

$$\begin{aligned} u'(t) &= -v(t) - \lambda\, u(t)(1 - u^2(t) - v^2(t)), \\ v'(t) &= u(t) - \theta\, \lambda\, v(t)(1 - u^2(t) - v^2(t)). \end{aligned} \tag{11}$$

When the integration starts for $t = 0$ on the unit circle, i.e., with $u^2(0) + v^2(0) = 1$, then the resulting solution $(u(t),v(t))$ oscillates along the unit circle, with $u^2(t) + v^2(t) = 1$ for all t.

In Eq. (11), the parameter $\lambda << 0$ characterizes the degree of stiffness, while the (moderate-sized) parameter θ, if chosen as $\theta \neq 1$, causes an additional non-normality effect for the Jacobian of the system along the smooth solution. Fig. 3 shows a phase portrait of the general behavior of solutions to Eq. (11). For preparing this visualization we have used the computer algebra system Maple ([10]); the problem parameters are chosen as $\lambda = -10$ and $\theta = 3$.

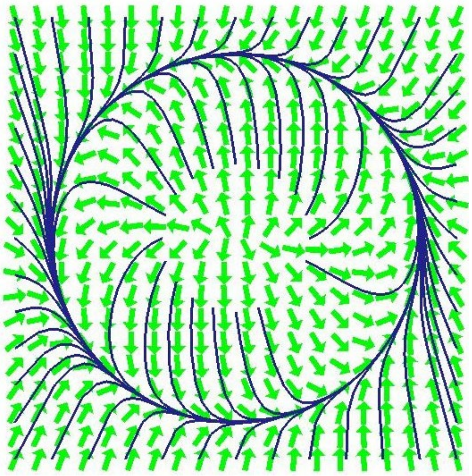

Fig. 3: Phase portrait *(u,v)* for example from Eq. (11)

Note that $(u,v) = (0,0)$ is a repelling fixed point. All transient solutions $(u(t),v(t))$, either from inside or from outside, rapidly converge towards the smooth solution following the unit circle. So this is a typical stiff situation. According to conventional folklore, the stiffness of an ODE system is characterized by the eigenvalue distribution of the Jacobian of its defining vector field. For the example from Eq. (11), these eigenvalues show such a typical behavior very close to the unit circle, but especially inside the unit circle, where transient solutions strongly increase, this is not the case and the eigenvalues are strictly positive, see [9].

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