Some new results for Hasimoto surfaces

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***Abstract* – Let σ=σ(s,t) be the position vector of a curve Γ moving on surface M in E^3 such that σ=σ(s,t) is a unit speed curve for all t. If the surface M is a Hasimoto surface, then, the position vector σ satisfy the following condition**

$$\sigma_t = \sigma_s \wedge \sigma_{ss}$$

also called as smoke ring equation or vortex filament [1]. In that work, we investigate the geometric properties according to Bishop frame of Hasimoto surfaces in Euclidean 3-space. Also, we give some characterization of parameter curves given according to Bishop frame of Hasimoto surfaces.

Keywords – Hasimoto Surface, Euclidean Space, vortex filament, Bishop frame, smoke ring equation.

Introduction

In [8], Da Rios invoked what is now known as the localized induction approximation to derive a pair of coupled nonlinear equations guiding the time evolution of the torsion and curvature of vortex filament also called smoke ring equations. Additively In 1972, Hasimoto [2] demonstrated that the Da Rios equations may be associated to generate the celebrated nonlinear Schrodinger (NLS) equation of soliton theory and also in this work, he considered a proximity to the selfinduced motion of a thin isolated vortex filament moving without extending in an incompressible fluid. Finally he obtain that if the position vector of vortex filament is σ=σ(s,t), then the formula

$$\sigma_t = \sigma_s \wedge \sigma_{ss}$$

is hold. In [8], the Da Rios equations and their composition, the NLS equation, are derived in a purely geometric manner via a binormal motion of an inextensible curve. In [3], authors discussed on the Hasimoto surface in E^3_1, where they invastiged its geometric properties and also gave some characterizations of parametric curves of this surface.

In that work, we move the study of Hasimoto surfaces started in [3] into the Minkowski space. First, we investigate the geometric properties according to Bishop frame of Hasimoto surfaces in Euclidean 3-space. Finally, we give some characterization of parameter curves given according to Bishop frame of Hasimoto surfaces.

Preliminiaries

Let E^3 denote the three-dimensional Euclidean space, that is, the real vector space R^3 endowed with the Riemann metric

$$<,> = (d\xi_0)^2 + (d\xi_1)^2 + (d\xi_2)^2$$

where (ξ_0, ξ_1, ξ_2) is rectangular coordinate system of E^3. Let u an arbitrary vector in E^3. So, the norm of u is given by $\|u\| = \sqrt{|\langle u,u\rangle|}$, [7].

Let $\mathbb{III}$ be a simply-connected domain in $E^2(t; s)$ and $\boldsymbol{\sigma}: \mathbb{III} \to \boldsymbol{E^3}$ an immersion in E^3. If σ=σ(s,t) is a parametrization of surface M in E^3, then the unit normal vector field N on M is given by

$$N = \frac{\sigma_s \wedge \sigma_t}{\|\sigma_s \wedge \sigma_t\|}$$

where $\sigma_s = \partial\sigma/\partial s$, and $\sigma_t = \partial\sigma/\partial t$, $\wedge$ stands for the Euclidean cross product of $\boldsymbol{E}^3$ [7].

The metric <,> on each tangent plane of M is determined by the first fundemantel form

$$I = < d\sigma, d\sigma > = Eds^2 + 2Fdsdt + Gdt^2$$

with differentiable coefficients

$$E = \langle \sigma_s, \sigma_s \rangle, F = \langle \sigma_s, \sigma_t \rangle, G = \langle \sigma_t, \sigma_t \rangle$$

Since we have,

$$detI = EG - F^2$$

The shape operatör of the immersion is indicated by the second fundamental form

$$II = -\langle dN, d\sigma \rangle = eds^2 + 2fdsdt + gdt^2$$

with differentiable coefficients

$$e = \langle \sigma_{ss}, N \rangle, f = \langle \sigma_{st}, N \rangle, g = \langle \sigma_{tt}, N \rangle.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve Γ has vanishing second derivative. One can state parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame [5]. The tangent vector and any convenient arbitrary basis for remainder of the frame are used [5,6]. The Bishop frame is expressed as;

$$\begin{bmatrix} t \\ y \\ z \end{bmatrix}_s = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ z \end{bmatrix}, \qquad (1)$$

where the set of {t,y,z} is called as Bishop trihedra and the functions k_1 and k_2 are the Bishop curvatures (see for details in [4]).

Main results

In this section, as we mentioned before, we move the study of Hasimoto surfaces started in [3] into the Minkowski space. So, we would like to give our main aim as following:

Main theorem: Let σ=σ(s,t) be the position vector of a curve Γ moving on surface M in Euclidean 3-space such that σ=σ(s,t) is a unit speed curve for all t. Then the derivatives of followings are satisfied;

$$\begin{bmatrix} t \\ y \\ z \end{bmatrix}_t = \begin{bmatrix} 0 & -(k_2)_s & (k_1)_s \\ (k_2)_s & 0 & -\frac{k_1^2+k_2^2}{2} \\ -(k_1)_s & \frac{k_1^2+k_2^2}{2} & 0 \end{bmatrix} \begin{bmatrix} t \\ y \\ z \end{bmatrix} \quad (2)$$

where {t,y,z} is the Bishop frame field and the functions k_1 and k_2 are the Bishop curvature functions of the curve Γ for all t.

Proof. We would like to obtain time derivatives of the Bishop frame {t,y,z} which is given the form

$$t_t = \alpha y + \beta z,$$
$$y_t = -\alpha t + \gamma z,$$
$$z_t = -\beta t - \gamma y, \quad (3)$$

where α, β and γ are smooth functions. Fort his aim, we have to find $\{\alpha, \beta, \gamma\}$ non-zero smooth functions s in terms of the Bishop curvatures k_1 and k_2. Using compatibility conditions $t_{ts} = t_{st}$, (3) yield

$$\alpha_s = (k_1)_t - k_2\gamma,$$
$$\beta_s = (k_2)_t + k_1\gamma$$
$$\gamma_s = -k_1\beta - k_2\alpha.$$

Now, we assume that the velocity of the curve is of the form

$$\sigma_t = \lambda t + \mu y + \vartheta z.$$

On the other hand from imposition of condition $\sigma_{st} = \sigma_{ts}$, we find the following equalities

$$\lambda_s = k_1\mu + k_2\vartheta,$$
$$\alpha = k_1\lambda + \mu_s,$$
$$\beta = k_2\lambda + \vartheta_s, \quad (4)$$

One can choose the correspondance fort he surface M as $\{\lambda, \mu, \vartheta\} \to \{0, -k_2, k_1\}$. Thus, the velocity vector is given by

$$\sigma_t = \sigma_s \times \sigma_{ss} = -k_2 y + k_1 z$$

which is solution of smoke ring equation. Hence, we can rewrite $(4)_{2,3}$ under the correspondance as

$$\alpha = -(k_2)_s, \beta = (k_1)_s.$$

Substituting the last equations into $(4)_1$ gives

$$\gamma = -\frac{k_1^2 + k_2^2}{2}.$$

Thus, the proof of main theorem is completed.

Some Characterization of Parameter Curves of Hasimoto surfaces

In this section, we would like to give new characterizations of parameter curves of Hasimoto surfaces in Euclidean 3-spaces.

Theorem. Assume σ=σ(s,t) is a Hasimoto surface in E^3. Then the followings are satisfied;

i. s-parameter curves of the surface σ=σ(s,t) are geodesics,
ii. t-parameter curves of the surface σ=σ(s,t) are geodesics if and only if

$$k_1(k_1)_t + k_2(k_2)_t = 0$$

where, k_1 and k_2 are Bishop curvature functions of the curve for all t.

Theorem. Assume σ=σ(s,t) is a Hasimoto surface in E^3. Then the followings are satisfied;

i. s-parameter curves of the surface σ=σ(s,t) are asymtotics if and only if $\kappa = 0$,
ii. t-parameter curves of the surface σ=σ(s,t) are asymptotics if and only if

$$2(k_2(k_1)_t + k_1(k_2)_t) = (k_1^2 + k_2^2)^2$$

where, k_1 and k_2 are Bishop curvature functions of the curve for all t.

Corollary. If s-parameter curves of a Hasimoto surface σ=σ(s,t) in E^3 are asymptotics, then the t-parameter curves are also asymptotics.

Corollary. The parameter curves of a Hasimoto surface σ=σ(s,t) in E^3 are lines of curvature if and only if

$$k_1(k_1)_s + k_2(k_2)_s = 0.$$

Conclusion

In this paper we studied the Hasimoto surfaces in Euclidean 3-spaces. Also we obtained the time derivatives of Bishop trihedra {t,y,z} of the curve moving on Hasimoto surfaces. After, we obtained some characterizations of parameter curves of Hasimoto surfaces in E^3.

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